Journal of Combinatorial Theory, Series B **74**, 1–7 (1998)
Article No. TB981823

On k-Critical Connected Line Graphs

Matthias Kriesell*

TU Berlin, FB Mathematik, D-10623 Berlin, Germany

Received August 16, 1996

We show that any line graph contains a set of three vertices which is not included in a smallest separating vertex set. This was conjectured by Maurer and Slater.

the set of all smallest separating sets will be denoted by $\mathcal{T}_G$. G is called *k-critical* if each set of at most k vertices is included in some $T \in \mathcal{T}_G$. Note that a k-critical graph is not complete. If G is a k-critical graph then $k \leqslant \kappa(G)/2$; this was conjectured by Slater in [4] and proved by Su in [7]. Earlier, Hamidoune proved this conjecture for line graphs [1]. We shall prove that $k \leqslant 2$ if G is a k-critical line graph.

An edge connecting two vertices x and y of a graph $G=(V, E)$ will be denoted by $[x, y]$. For $X \subseteq V(G)$ we define $N_G(X) := \{ y \in V(G)$: there is an $x \in X$ for which $[x, y] \in E(G)\} - X$ and write $N_G(x)$ instead of $N_G(\{x\})$, $x \in V(G)$. $\gamma_G(x) := |N_G(x)|$ denotes the *degree* of x in G. The index G will be left out if it is clear from the context. $G(X)$ denotes the subgraph of G induced by the vertex set $X \subseteq V(G)$. We call X *connected* if $G(X)$ is connected and we call X *complete* if $G(X)$ is complete. K_n denotes the complete graph on n vertices. An induced subgraph of G on four vertices is a *claw* of G if it contains one vertex of degree 3 and three of degree 1.

$\mathcal{P}(M)$ denotes the *power set* of a set M. Let $\mathcal{S} \subseteq \mathcal{P}(V(G))$. Let $T \in \mathcal{T}_G$ and suppose $S \subseteq T$ for some $S \in \mathcal{S}$. The union of at least one but not of all components of $G-T$ is called a *$T-\mathcal{S}$-fragment* of G. An *$\mathcal{S}$-fragment* is a $T-\mathcal{S}$-fragment for a $T \in \mathcal{T}_G$. For each $T-\mathcal{S}$-fragment F we define the $T-\mathcal{S}$-fragment $\bar{F}$ by $\bar{F} := G-(T \cup F)$. An $\mathcal{S}$-fragment of minimum cardinality is called an *$\mathcal{S}$-atom* of G, a *$T-\mathcal{S}$-atom* A is an $\mathcal{S}$-atom with $N(A)=T$. In case of $\mathcal{S} = \{\varnothing\}$ we omit the reference to $\mathcal{S}$.

Repeating the definition of [3], we call G *$\mathcal{S}$-critical* if $\mathcal{S} \neq \varnothing$, for each $S \in \mathcal{S}$ there is a $T \in \mathcal{T}_G$ with $S \subseteq T$, and for each $T-\mathcal{S}$-fragment F

* Supported by the Deutsche Forschungsgemeinschaft Grant GRK 219/2-96.

0095-8956/98 $25.00

there are $S' \in \mathcal{S}$ and $T' \in \mathcal{T}_G$ satisfying $S' \subseteq T' - \bar{F}$ and $F \cap T' \neq \emptyset$. For example, G is k-critical if and only if it is $\{X \subseteq V(G) : |X| \leqslant k\}$-critical.

The following theorem describes two important properties of $\mathcal{S}$-atoms in $\mathcal{S}$-critical graphs. In order to make this paper self-contained we give a short proof, which does not rely on the difficult results of [3].

THEOREM 1 [3, Theorem 1]. *Let A be a $T_A - \mathcal{S}$-atom of G. Let there be $S' \in \mathcal{S}$ and $T' \in \mathcal{T}_G$ satisfying $S' \subseteq T' - \bar{A}$ and $A \cap T' \neq \emptyset$. Then $A \subseteq T'$ holds and $|A| \leqslant |T_A - T'|/2$.*

Proof. Consider two arbitrary fragments F, F^*, and let $T := N(F)$, $T^* := N(F^*)$. Suppose that $F \cap F^* \neq \emptyset$; since $N(F \cap F^*) \subseteq (F \cap T^*) \cup (T - \overline{F^*})$, we obtain $\kappa(G) \leqslant |F \cap T^*| + |T| - |T \cap \overline{F^*}|$, and therefore,

$$\begin{aligned}&|F \cap T^*| \geqslant |\overline{F^*} \cap T| \text{ and}\\&\text{if equality holds then } F \cap F^* \text{ is a fragment with}\\&N(F \cap F^*) = (F \cap T^*) \cup (T - \overline{F^*}).\end{aligned} \tag{1}$$

In particular, $F \cap F^*$ and $\bar{F} \cap \overline{F^*}$ are both fragments if and only if they are both nonempty.

Take an arbitrary T'-fragment F'. Since $S' \subseteq T'$, we have $|F'|, |\overline{F'}| \geqslant |A|$.

Assume for a while that $A \cap F' \neq \emptyset$. If $|A \cap T'| = |\overline{F'} \cap T_A|$ then $A \cap F'$ is a fragment by (1); since $S' \subseteq N(A \cap F')$, it is even an $\mathcal{S}$-fragment — which is a contradiction, since $A \cap F'$ contains less vertices than A. Thus, by (1), $|A \cap T'| > |\overline{F'} \cap T_A|$ and $\bar{A} \cap \overline{F'} = \emptyset$. If $A \cap \overline{F'} = \emptyset$ then we conclude $|\overline{F'} \cap T_A| = |\overline{F'}| \geqslant |A| \geqslant |A \cap T'|$, a contradiction. Thus, $A \cap \overline{F'} \neq \emptyset$. Applying similar arguments to $\overline{F'}$, we conclude $|\overline{F'} \cap T_A| > |\bar{A} \cap T'|$ and $\bar{A} \cap F' = \emptyset$; therefore, $|\bar{A} \cap T'| = |\bar{A}| \geqslant |A| > |A \cap T'| > |\overline{F'} \cap T_A| > |\bar{A} \cap T'|$, a contradiction.

Thus, we found $A \cap F' = A \cap \overline{F'} = \emptyset$, which implies $A \subseteq T'$. If $F' \nsubseteq T_A$ then $F' \cap \bar{A} \neq \emptyset$ and thus, by (1), $|F' \cap T_A| \geqslant |A|$. Since the latter inequality is clearly true if $F' \subseteq T_A$, and since it does also hold if we replace F' by $\overline{F'}$, we obtain $|A| \leqslant (|F' \cap T_A| + |\overline{F'} \cap T_A|)/2 = |T_A - T'|/2$. ∎

In order to characterize the line graphs of *multigraphs* we introduce the following concept. A graph $\underline{G}$ is a *partition graph* of a graph G if the following conditions are satisfied:

(P1) $V(\underline{G})$ is a partition of $V(G)$.

(P2) For $\underline{x}, \underline{y}$ in $V(\underline{G})$ we have $[\underline{x}, \underline{y}] \in E(\underline{G})$ if and only if $N_G(\underline{x}) \cap \underline{y} \neq \emptyset$.

(P3) $\underline{x}$ is complete for each $\underline{x} \in V(\underline{G})$.

(P4) $\underline{x} \cup \underline{y}$ is complete for each $[\underline{x}, \underline{y}] \in E(\underline{G})$.

Let $\underline{G}$ be a partition graph of G. For $x \in V(G)$ let $\underline{x}_{\underline{G}}$ denote the class of $V(\underline{G})$ which contains x. For $X \subseteq V(G)$ we define $\underline{X}_{\underline{G}} := \{\underline{x}_{\underline{G}}: x \in X\}$. From the properties above it follows immediately that, for every smallest separating set T of G, $\underline{x}_{\underline{G}} \subseteq T$ if and only if $x \in T$. Again, we leave out the index $\underline{G}$ if it is clear from the context. Keep in mind that the symbol $\underline{x}$ denotes both a *subset of vertices* of G and a *vertex* of $\underline{G}$.

A triangle Δ of a graph G is called *odd* if there is a vertex in G adjacent to exactly one vertex or to all vertices of Δ. Otherwise, we will call Δ an *even* triangle. The following characterization of line graphs of graphs in terms of claws and odd triangles was first discovered by Van Rooij and Wilf [6]:

THEOREM 2. *A graph G is the line graph of a graph if and only if G is claw free and two odd triangles with exactly one common edge induce a K_4.*

Since G is the line graph of a *multigraph* if and only if there exists a partition graph of G which is the line graph of a *graph*, we obtain a characterization of the line graphs of multigraphs.

THEOREM 3. *A graph G is the line graph of a multigraph if and only if there is a claw free partition graph $\underline{G}$ of G where two odd triangles with exactly one common edge induce a K_4.*

To prepare the proof of the main result, we shall state the next three lemmas. Obviously, a noncomplete neighborhood of a vertex of a line graph can be partitioned into two complete vertex sets; Lemma 1 describes properties of edges between them. Lemma 2 and Lemma 3 deal with the structure of small fragments in line graphs.

LEMMA 1. *Let G be the line graph of a graph and $z \in V(G)$. Let $N(z)$ be noncomplete and partitioned into two complete vertex sets X, Y of cardinality at least 3 each.*

1. *Then two distinct edges between X and Y have no vertex in common.*

2. *z and the vertices of an edge between X and Y induce an even triangle.*

Proof. Assume that $x \in X$ has two distinct neighbors $y, y' \in Y$. Since y, y' form an odd triangle together with x and, also together with an arbitrary $y'' \in Y - \{y, y'\}$, x must be adjacent to *all* vertices of Y. Since $|X|, |Y| \geqslant 3$, x, z form an odd triangle together with each $u \in N(z) - \{x\}$. Thus, $N(z)$ must be complete, which is a contradiction. This proves 1.

Take $x \in X$, $y \in N(x) \cap Y$, and assume that x, y, z form an odd triangle. By 1, there cannot exist a vertex which is adjacent to *all* three vertices x, y, and z. Since every vertex in $N(z)$ is adjacent to x or y, there must be a vertex u in $N(x) - (N(z) \cup N(y))$ or in $N(y) - (N(z) \cup N(x))$. Without loss of generality, the latter case occurs. For all $y' \in Y - \{y\}$ the set $\{x, y, y', u\}$ does not induce a claw; since 1 holds, there must be an edge $[u, y']$ for all $y' \in Y$. Since $|Y| \geqslant 3$, $\{u, y, y'\}$ and $\{z, y, y'\}$ induce odd triangles; this implies $[u, z] \in E(G)$, a contradiction. ▮

LEMMA 2. *Let $\underline{G}$ be a partition graph of G and let F be a T-fragment of G.*

1. *Let $\underline{G}$ be claw free, $x \neq y$ in F, and $N(x) \cap N(y) \cap T \neq \emptyset$. Then $[x, y] \in E(G)$.*
2. *Let $\underline{G}$ be the line graph of a graph, x, y in F, $\underline{x} \neq \underline{y}$, and $t \neq u$ in $N(x) \cap N(y) \cap T$. Then $[t, u] \in E(G)$.*
3. *Let $\underline{G}$ be claw free and $|F| \leqslant \kappa(G)/2$. Then F is complete.*

Proof. For proving 1 let $z \in N(x) \cap N(y) \cap T$ and take an arbitrary $u \in N(z) \cap \bar{F}$. Since $\{x, y, z, u\}$ does not induce a claw, $[x, y] \in E(G)$ follows.

For proving 2 we may assume that $\underline{t} \neq \underline{u}$. Then $\underline{G}(\{\underline{x}, \underline{y}, \underline{t}\})$, $\underline{G}(\{\underline{x}, \underline{y}, \underline{u}\})$ are distinct triangles in $\underline{G}$ by 1; since the vertices $\underline{t}, \underline{u}$ both have a neighbor in $V(\underline{G}) - (\underline{F} \cup \underline{T})$, which is neither adjacent to $\underline{x}$ nor is adjacent to $\underline{y}$ in $\underline{G}$, these triangles must be odd. Thus, $[\underline{t}, \underline{u}] \in E(\underline{G})$, which implies $[t, u] \in E(G)$.

For proving 3, note that, since $|F| \leqslant \kappa(G)/2$, any two vertices in F must have a common neighbor in T and thus must be adjacent by 1. ▮

LEMMA 3. *Let $\underline{G}$ be a partition graph of G and let $\underline{G}$ be the line graph of a graph. Let F be a T-fragment of G, $|F| \leqslant \kappa(G)/2$, and suppose that $\underline{F}$ does not induce an even triangle in $\underline{G}$. Then each vertex of $\underline{T}$ is adjacent to exactly one or to all vertices in $\underline{F}$.*

Proof. This is clearly true if $|\underline{F}| \leqslant 2$. Let us assume that $|\underline{F}| \geqslant 3$. By Lemma 2, $\underline{F}$ must be complete. It follows that any three vertices of $\underline{F}$ induce an odd triangle in $\underline{G}$. Consider a vertex $\underline{t} \in \underline{T}$; if $\underline{t}$ has two neighbors $\underline{f} \neq \underline{f}'$ in $\underline{F}$ then the set $\{\underline{f}, \underline{f}', \underline{t}\}$ induces an *odd* triangle in $\underline{G}$ (since $\underline{t}$ has a neighbor in $V(\underline{G}) - (\underline{F} \cup \underline{T})$), and, therefore, $\underline{t}$ must be adjacent to all vertices in $\underline{F}$. ▮

Now we are prepared to prove the conjecture mentioned in [5, Conjecture 3.11]. It is an immediate consequence of our main result.

THEOREM 4. *Let G be the line graph of a multigraph and let G be noncomplete and connected. Then G contains a set of three connected vertices which is not contained in a smallest separating set.*

Proof. Assume the contrary. Then there exists a noncomplete connected line graph G of a multigraph where any three connected vertices are contained in a smallest separating set. Choose G with minimum number of vertices.

Let $\underline{G}$ be a partition graph of G and let $\underline{G}$ be the line graph of a graph. We take $x \in V(G)$ with $|\underline{x}| \geqslant |\underline{y}|$ for all $y \in V(G)$. Let

$$\mathscr{S} := \{\{x, y, z\} : \underline{y} \neq \underline{z} \text{ in } V(\underline{G}) - \{\underline{x}\},$$
$$[\underline{x}, \underline{y}] \in E(\underline{G}),\ [\underline{y}, \underline{z}] \in E(\underline{G}) \text{ and } [\underline{x}, \underline{z}] \notin E(\underline{G})\}.$$

First we prove that G is $\mathscr{S}$-critical.

1. For proving $\mathscr{S} \neq \emptyset$ it suffices to show that $\underline{x}$ is not adjacent to all $\underline{y} \in V(\underline{G})$. If we had $N_{\underline{G}}(\underline{x}) = V(\underline{G}) - \{\underline{x}\}$ then $G - \underline{x}$ must be a noncomplete line graph of a multigraph whose smallest separating sets are precisely the sets $T - \underline{x}$ with $\underline{x} \subseteq T \in \mathscr{T}_G$. Thus, any three connected vertices of $G - \underline{x}$ are contained in a smallest separating set of $G - \underline{x}$. Since $G - \underline{x}$ is noncomplete, x has two distinct neighbors u, v in $V(G) - \underline{x}$. Since u, v, x are contained in a smallest separating set of G, it follows $\kappa(G) > |\underline{x}|$; consequently, $G - \underline{x}$ must be connected. Thus, $G - \underline{x}$ is a noncomplete connected line graph of a multigraph where any three connected vertices are contained in a smallest separating set—contradicting the choice of G.

2. For any given $S \in \mathscr{S}$ our assumption guarantees the existence of a $T \in \mathscr{T}_G$ with $S \subseteq T$ (moreover, $\bigcup \underline{S} \subseteq T$).

3. Let F be a $T - \mathscr{S}$-fragment, say $S = \{x, y, z\} \subseteq T$, where $\underline{y} \neq \underline{z}$ in $V(\underline{G}) - \{\underline{x}\}$, $[\underline{x}, \underline{y}] \in E(\underline{G})$, $[\underline{y}, \underline{z}] \in E(\underline{G})$ and $[\underline{x}, \underline{z}] \notin E(\underline{G})$. Take a chordless x, z-path with all internal vertices contained in F, say $x =: x_1, x_2, ..., x_n := z$, $n \geqslant 3$. Then we have $S' := \{x_1, x_2, x_3\} \in \mathscr{S}$, $S' \cap F \neq \emptyset$, and $S' \cap \bar{F} = \emptyset$. This implies the existence of a $T' \in \mathscr{T}_G$ with $T' \supseteq S'$ and $T' \cap F \neq \emptyset$.

Take a $T - \mathscr{S}$-atom A, say $S = \{x, y, z\} \subseteq T$, where $\underline{y} \neq \underline{z}$ in $V(\underline{G}) - \{\underline{x}\}$, $[\underline{x}, \underline{y}] \in E(\underline{G})$, $[\underline{y}, \underline{z}] \in E(\underline{G})$ and $[\underline{x}, \underline{z}] \notin E(\underline{G})$. By Theorem 1, we have $|A| \leqslant \kappa(G)/2$. We shall prove:

$$\text{The subgraph induced in } \underline{G} \text{ by the vertex set } \underline{A} \cup \underline{T} \subseteq V(\underline{G}) \text{ contains an even triangle.} \tag{2}$$

If $|\underline{A}| = 1$ then $\underline{A} = \{\underline{a}\}$ for some $\underline{a} \in V(\underline{G})$; suppose that there is a partition of $N_{\underline{G}}(\underline{a})$ into two complete vertex sets such that one of them consists of at most two vertices $\underline{x_1}, ..., \underline{x_n}$, $n \leqslant 2$; then there would be a $T' \in \mathcal{T}_G$ containing $a, x_1, ..., x_n$ and thus containing $\underline{a} \cup \underline{x_1} \cup \cdots \cup \underline{x_n}$, which implies that $N_G(\underline{a}) - T'$ is complete, a contradiction. Thus, $N_{\underline{G}}(\underline{a})$ is noncomplete and partitionable into two complete vertex sets of cardinality at least 3 each. By Lemma 1, $\underline{G}(\{\underline{a}, \underline{x}, \underline{y}\})$ or $\underline{G}(\{\underline{a}, \underline{y}, \underline{z}\})$ is an *even* triangle.

Therefore we may assume that $|\underline{A}| \geqslant 2$. Moreover, we may suppose that $\underline{A}$ does not induce an even triangle in $\underline{G}$.

Let $D := \{d \in T\colon |N_{\underline{G}}(\underline{d}) \cap \underline{A}| = 1\}$. By Lemma 3, each vertex of $T - D$ is adjacent to all vertices of A. By Lemma 2, $A \cup (T - D)$ is complete.

Let $g\colon A \to \mathcal{P}(D)$ defined by $g(a) := N_G(a) \cap D$.

Fix $a \in A$ for a while and let $\underline{g(a)} = \{\underline{d_1}, ..., \underline{d_n}\}$ for certain representative vertices $d_1, ..., d_n \in D$.

Observe that $n \geqslant 3$, for otherwise there would be a $T' \in \mathcal{T}_G$ containing $a, d_1, ..., d_n$ and thus containing $\underline{a} \cup \underline{d_1} \cup \cdots \cup \underline{d_n}$. Then $N_G(\underline{a}) - T'$ is complete as a subset of $A \cup (T - D)$, which is impossible.

Furthermore, we have $|\underline{d_1} \cup \cdots \cup \underline{d_n}| \leqslant |\underline{a}|$ (for otherwise we had $|N_G(A - \underline{a})| = |(T \cup \underline{a}) - (\underline{d_1} \cup \cdots \cup \underline{d_n})| < |T|$, a contradiction). This implies, in particular, $T - D \neq \emptyset$.

Thus we have proved that $|\underline{d_1}| + \cdots + |\underline{d_n}| \leqslant |\underline{a}|$. Since $n \geqslant 3$ and $|\underline{d_i}| \geqslant 1$ for all $i \in \{1, ..., n\}$, we obtain $|\underline{d}| < |\underline{a}|$ for each $a \in A$ and for each $d \in g(a)$. It follows $x \notin D$, and hence $z \in D$.

Suppose $y \in T - D$ and take a vertex $\underline{a} \in N_{\underline{G}}(\underline{z}) \cap \underline{A}$; since $[\underline{x}, \underline{z}] \notin E(\underline{G})$, one of the triangles $\underline{G}(\{\underline{y}, \underline{a}, \underline{x}\})$, $\underline{G}(\{\underline{y}, \underline{a}, \underline{z}\})$ has to be even.

Suppose $y \in D$ and take a vertex $\underline{a} \in N_{\underline{G}}(\underline{y}) \cap \underline{A}$, $\underline{b} \in \underline{A} - \{\underline{a}\}$; since $[\underline{y}, \underline{b}] \notin E(\underline{G})$, one of the triangles $\underline{G}(\{\underline{x}, \underline{a}, \underline{y}\})$, $\underline{G}(\{\underline{x}, \underline{a}, \underline{b}\})$ has to be even.

This proves (2).

Thus, there exists an even triangle in $\underline{G}$. Let $\{\underline{a}, \underline{b}, \underline{c}\}$ be its vertex set and take a smallest separating set T of G containing a, b, c and therefore containing $\underline{a} \cup \underline{b} \cup \underline{c}$.

For $d \neq e$ in $\{a, b, c\}$ define $N_{de} := N_G(d) \cap N_G(e) - (\underline{a} \cup \underline{b} \cup \underline{c})$. $\{N_{ab}, N_{ac}, N_{bc}\}$ is a partition of $N_G(\underline{a} \cup \underline{b} \cup \underline{c})$ into complete vertex sets (since $\underline{G}(\{\underline{a}, \underline{b}, \underline{c}\})$ is even). Take a T-fragment F of G; since $N_G(a) \cap F \neq \emptyset$ and $N_G(a) \cap \bar{F} \neq \emptyset$ and $N_G(a) - (\underline{a} \cup \underline{b} \cup \underline{c}) = N_{ab} \cup N_{ac}$, we may assume $N_{ab} \cap F \neq \emptyset$ and $N_{ac} \cap \bar{F} \neq \emptyset$ without loss of generality. This implies $N_{ab} \cap \bar{F} = N_{ac} \cap F = \emptyset$. Hence we have $N_{bc} \cap \bar{F} \neq \emptyset$ and $N_{bc} \cap F \neq \emptyset$, contradicting the fact that N_{bc} is complete.

This completes the proof of Theorem 4. ∎

Clearly, Theorem 4 implies

THEOREM 5. *There is no* 3-*critical line graph.*

The graphs $K_{2k+2} - (1 - factor)$ are k-critical $2k$-connected graphs and show that our result cannot be generalized to claw-free graphs. To characterize all 2-critical line graphs should be a hard task; it is well known that $\kappa(G) \geqslant 4$ for each 2-critical graph G [2]. The 2-critical graphs of connectivity 4 are all known, and among them there are exactly three line graphs, namely $L(K_4)$, $L(K_{3,3})$, and $L(K_{4,4} - (1 - factor))$ (see [2]). The line graphs of squares of cycles show that there are 2-critical 6-connected line graphs of arbitrary diameter.

REFERENCES

1. Y. O. Hamidoune, On multiply critically h-connected graphs, *J. Combin. Theory Ser. B* **30** (1981), 108–112.
2. W. Mader, On k-critically n-connected graphs, *Progr. in Graph Theory* (1984), 389–398.
3. W. Mader, Generalizations of critical connectivity of graphs, *Discrete Math.* **72** (1988), 267–283.
4. St. B. Maurer and P. J. Slater, On k-critical, n-connected graphs, *Discrete Math.* **20** (1977), 255–262.
5. St. B. Maurer and P. J. Slater, On k-minimally n-edge connected graphs, *Discrete Math.* **24** (1978), 185–195.
6. A. van Rooij and H. Wilf, The interchange graph of a finite graph, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 263–269.
7. J. Su, On locally k-critically n-connected graphs, *Discrete Math.* **120** (1993), 183–190.